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**Monday 19 October 2020 – Afternoon**

**A Level Mathematics B (MEI)**

**H640/03 Pure Mathematics and Comprehension**

**Time allowed: 2 hours**



**You must have:**

- the Printed Answer Booklet
- the Insert
- a scientific or graphical calculator

**INSTRUCTIONS**

- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give your final answers to a degree of accuracy that is appropriate to the context.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

**INFORMATION**

- The total mark for this paper is **75**.
- The marks for each question are shown in brackets [ ].
- This document has **12** pages.

**ADVICE**

- Read each question carefully before you start your answer.

## Formulae A Level Mathematics B (MEI) (H640)

### Arithmetic series

$$S_n = \frac{1}{2}n(a+l) = \frac{1}{2}n\{2a + (n-1)d\}$$

### Geometric series

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_\infty = \frac{a}{1-r} \text{ for } |r| < 1$$

### Binomial series

$$(a+b)^n = a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + \dots + {}^nC_r a^{n-r}b^r + \dots + b^n \quad (n \in \mathbb{N}),$$

$$\text{where } {}^nC_r = {}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots + \frac{n(n-1)\dots(n-r+1)}{r!}x^r + \dots \quad (|x| < 1, n \in \mathbb{R})$$

### Differentiation

| $f(x)$                   | $f'(x)$                          |
|--------------------------|----------------------------------|
| $\tan kx$                | $k \sec^2 kx$                    |
| $\sec x$                 | $\sec x \tan x$                  |
| $\cot x$                 | $-\operatorname{cosec}^2 x$      |
| $\operatorname{cosec} x$ | $-\operatorname{cosec} x \cot x$ |

$$\text{Quotient Rule } y = \frac{u}{v}, \frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

### Differentiation from first principles

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

### Integration

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$$

$$\int f'(x)(f(x))^n dx = \frac{1}{n+1}(f(x))^{n+1} + c$$

$$\text{Integration by parts } \int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

### Small angle approximations

$$\sin \theta \approx \theta, \cos \theta \approx 1 - \frac{1}{2}\theta^2, \tan \theta \approx \theta \text{ where } \theta \text{ is measured in radians}$$

**Trigonometric identities**

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \quad \left( A \pm B \neq \left(k + \frac{1}{2}\right)\pi \right)$$

**Numerical methods**

Trapezium rule:  $\int_a^b y \, dx \approx \frac{1}{2}h\{(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1})\}$ , where  $h = \frac{b-a}{n}$

The Newton-Raphson iteration for solving  $f(x) = 0$ :  $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

**Probability**

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cap B) = P(A)P(B|A) = P(B)P(A|B) \quad \text{or} \quad P(A|B) = \frac{P(A \cap B)}{P(B)}$$

**Sample variance**

$$s^2 = \frac{1}{n-1}S_{xx} \quad \text{where} \quad S_{xx} = \sum(x_i - \bar{x})^2 = \sum x_i^2 - \frac{(\sum x_i)^2}{n} = \sum x_i^2 - n\bar{x}^2$$

Standard deviation,  $s = \sqrt{\text{variance}}$

**The binomial distribution**

If  $X \sim B(n, p)$  then  $P(X = r) = {}^n C_r p^r q^{n-r}$  where  $q = 1 - p$

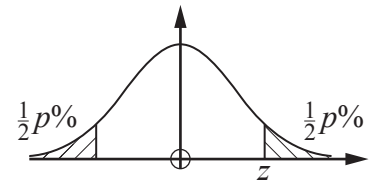
Mean of  $X$  is  $np$

**Hypothesis testing for the mean of a Normal distribution**

If  $X \sim N(\mu, \sigma^2)$  then  $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$  and  $\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$

**Percentage points of the Normal distribution**

|     |       |       |       |       |
|-----|-------|-------|-------|-------|
| $p$ | 10    | 5     | 2     | 1     |
| $z$ | 1.645 | 1.960 | 2.326 | 2.576 |

**Kinematics**

Motion in a straight line

$$v = u + at$$

$$s = ut + \frac{1}{2}at^2$$

$$s = \frac{1}{2}(u + v)t$$

$$v^2 = u^2 + 2as$$

$$s = vt - \frac{1}{2}at^2$$

Motion in two dimensions

$$\mathbf{v} = \mathbf{u} + \mathbf{a}t$$

$$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$

$$\mathbf{s} = \frac{1}{2}(\mathbf{u} + \mathbf{v})t$$

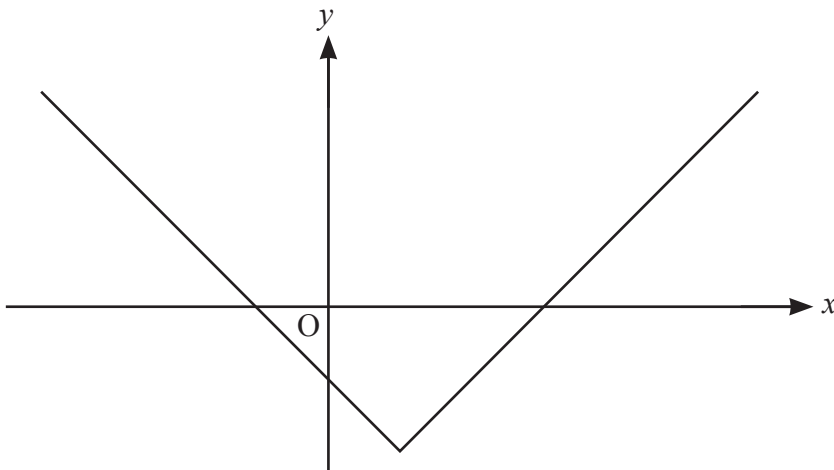
$$\mathbf{s} = \mathbf{v}t - \frac{1}{2}\mathbf{a}t^2$$

Answer **all** the questions.

**Section A** (60 marks)

1 Find the value of  $\sum_{r=1}^5 2^r(r-1)$ . [2]

2 The graph of  $y = |1-x|-2$  is shown in Fig. 2.



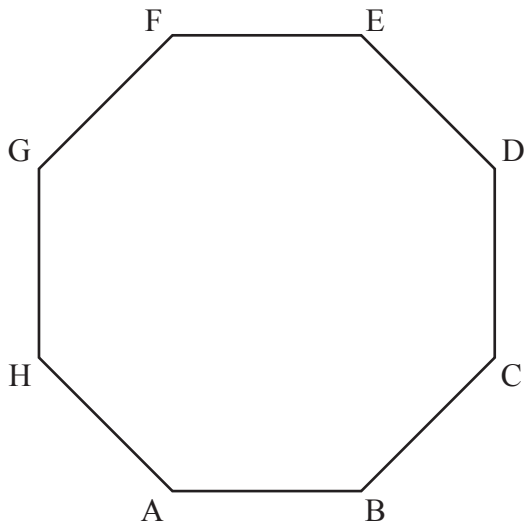
**Fig. 2**

Determine the set of values of  $x$  for which  $|1-x| > 2$ . [4]

3 A particular phone battery will last 10 hours when it is first used. Every time it is recharged, it will only last 98% of its previous time.

Find the maximum total length of use for the battery. [3]

4 Fig. 4 shows the regular octagon ABCDEFGH.



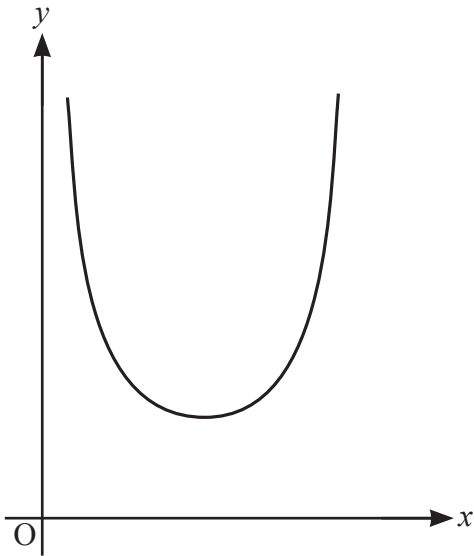
**Fig. 4**

$\vec{AB} = \mathbf{i}$ ,  $\vec{CD} = \mathbf{j}$ , where  $\mathbf{i}$  is a unit vector parallel to the  $x$ -axis and  $\mathbf{j}$  is a unit vector parallel to the  $y$ -axis.

Find an exact expression for  $\vec{BC}$  in terms of  $\mathbf{i}$  and  $\mathbf{j}$ .

[3]

5 Fig. 5 shows part of the curve  $y = \operatorname{cosec} x$  together with the  $x$ - and  $y$ -axes.



**Fig. 5**

- (a) For the section of the curve which is shown in Fig. 5, write down
- (i) the equations of the two vertical asymptotes, [2]
  - (ii) the coordinates of the minimum point. [1]
- (b) Show that the equation  $x = \operatorname{cosec} x$  has a root which lies between  $x = 1$  and  $x = 2$ . [2]
- (c) Use the iteration  $x_{n+1} = \operatorname{cosec}(x_n)$ , with  $x_0 = 1$ , to find
- (i) the values of  $x_1$  and  $x_2$ , correct to 5 decimal places, [1]
  - (ii) this root of the equation, correct to 3 decimal places. [1]
- (d) There is another root of  $x = \operatorname{cosec} x$  which lies between  $x = 2$  and  $x = 3$ .
- Determine whether the iteration  $x_{n+1} = \operatorname{cosec}(x_n)$  with  $x_0 = 2.5$  converges to this root. [1]
- (e) Sketch the staircase or cobweb diagram for the iteration, starting with  $x_0 = 2.5$ , on the diagram in the Printed Answer Booklet. [3]

- 6 (a) (i) Write down the derivative of  $e^{kx}$ , where  $k$  is a constant. [1]
- (ii) A business has been running since 2009. They sell maths revision resources online.

Give a reason why an exponential growth model might be suitable for the annual profits for the business. [1]

Fig. 6 shows the relationship between the annual profits of the business in thousands of pounds ( $y$ ) and the time in years after 2009 ( $x$ ). The graph of  $\ln y$  plotted against  $x$  is approximately a straight line.

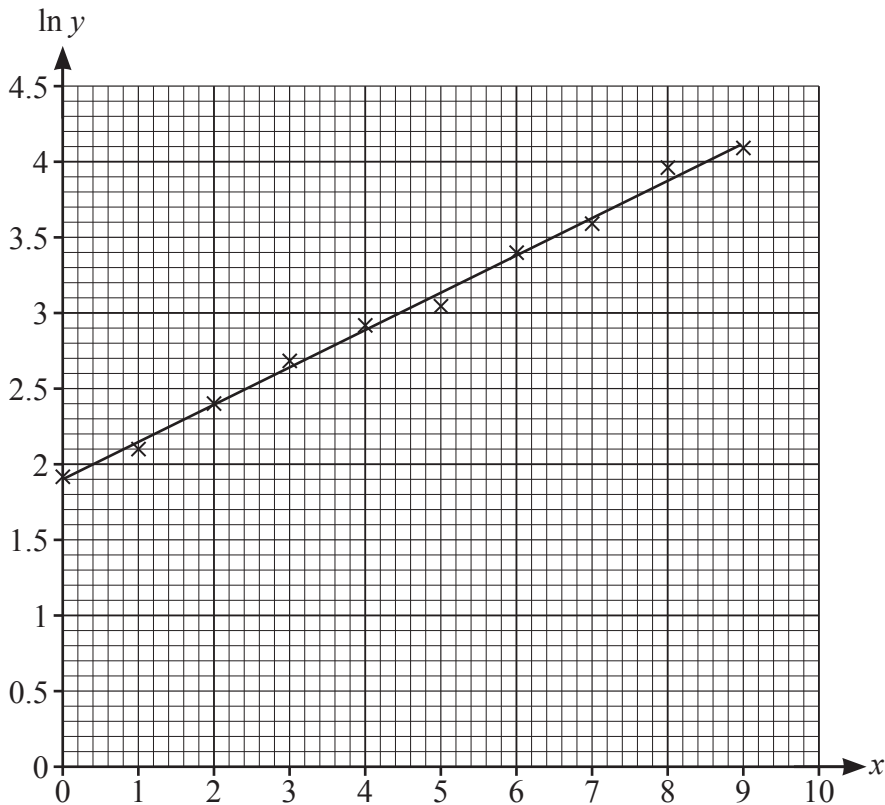


Fig. 6

- (b) Show that the straight line is consistent with a model of the form  $y = Ae^{kx}$ , where  $A$  and  $k$  are constants. [2]
- (c) Estimate the values of  $A$  and  $k$ . [4]
- (d) Use the model to predict the profit in the year 2020. [3]
- (e) How reliable do you expect the prediction in part (d) to be? Justify your answer. [1]

- 7 (a) Express  $\frac{1}{x} + \frac{1}{A-x}$  as a single fraction. [1]

The population of fish in a lake is modelled by the differential equation

$$\frac{dx}{dt} = \frac{x(400-x)}{400}$$

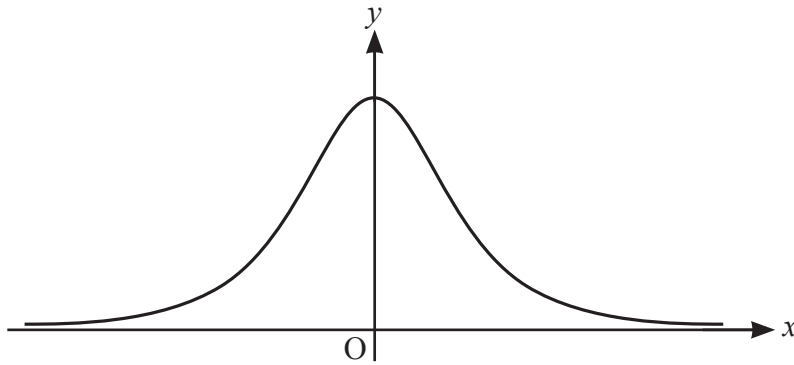
where  $x$  is the number of fish and  $t$  is the time in years.

When  $t = 0$ ,  $x = 100$ .

- (b) **In this question you must show detailed reasoning.**

Find the number of fish in the lake when  $t = 10$ , as predicted by the model. [8]

- 8 (a) The curve  $y = \frac{1}{(1+x^2)^2}$  is shown in Fig. 8.



**Fig. 8**

- (i) Show that  $\frac{d^2y}{dx^2} = \frac{20x^2 - 4}{(1+x^2)^4}$ . [5]

- (ii) **In this question you must show detailed reasoning.**

Find the set of values of  $x$  for which the curve is concave downwards. [3]

- (b) Use the substitution  $x = \tan \theta$  to find the exact value of  $\int_{-1}^1 \frac{1}{(1+x^2)^2} dx$ . [8]



Answer **all** the questions.

**Section B** (15 marks)

**The questions in this section refer to the article on the Insert. You should read the article before attempting the questions.**

9 (a) Show that if  $a = 1$  and  $b > 1$  then  $a^b < b^a$ . [2]

(b) Find integer values of  $a$  and  $b$  with  $b > a > 1$  and  $a^b$  not greater than  $b^a$  (a counter example to the conjecture given in lines 7–8). [1]

10 **In this question you must show detailed reasoning.**

Show that  $\int_e^\pi \frac{1}{x} dx = \ln \pi - 1$  as given in line 37. [2]

11 Show that  $e^x$  is an increasing function for all values of  $x$ , as stated in line 39. [2]

12 (a) Show that the only stationary point on the curve  $y = \frac{\ln x}{x}$  occurs where  $x = e$ , as given in line 45. [3]

(b) Show that the stationary point is a maximum. [3]

(c) It follows from part (b) that, for any positive number  $a$  with  $a \neq e$ ,

$$\frac{\ln e}{e} > \frac{\ln a}{a}.$$

Use this fact to show that  $e^a > a^e$ . [2]

**END OF QUESTION PAPER**





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## Which is bigger?

Which is bigger:  $\pi^e$  or  $e^\pi$ ? Using a calculator confirms that  $e^\pi$  is the larger, but how can this be proved without the use of a calculator?

### Simpler examples

It is often helpful in mathematics to consider simpler examples. It is easy to work out that  $3^4 > 4^3$ . In the expression  $3^4$ , 3 is the base and 4 is the exponent. Working with integers greater than 1, it is easy to find many examples where  $a^b > b^a$  if  $a < b$ . That is, using the smaller base and the larger exponent gives the larger result. This might lead us to conjecture that  $a^b > b^a$  if  $a < b$  and both  $a$  and  $b$  are integers greater than 1. However, it is also possible to find counter examples to this conjecture. 5

Exponents can also be rational numbers, and in general  $x^{\frac{p}{q}}$  denotes  $(\sqrt[q]{x})^p$  where  $p$  and  $q$  are integers and  $q$  is positive. So, any rational power of a positive number,  $x$ , can be defined. However, both  $e$  and  $\pi$  are irrational numbers. Considering the original question about  $\pi^e$  and  $e^\pi$  raises the issue of what is meant by an irrational power of a number. 10

### Extending the definition of power to irrational numbers

What, for example, is meant by  $2^\pi$ ? 15

An irrational number corresponds to a non-recurring infinite decimal. Rounding the decimal gives a rational approximation to the irrational number. For example, the following sequence gives increasingly accurate approximations to  $\pi$ .

3, 3.1, 3.14, 3.142, 3.1416, 3.14159, ...

Using a spreadsheet gives a sequence of approximations to  $2^\pi$ , as shown in Fig. C1. The limit of this sequence of approximations is the value of  $2^\pi$ . This limit cannot be evaluated with a spreadsheet but it is, in principle, possible to find the value to any required degree of accuracy. 20

|   | A       | B        |
|---|---------|----------|
| 1 | $k$     | $2^k$    |
| 2 | 3       | 8        |
| 3 | 3.1     | 8.574188 |
| 4 | 3.14    | 8.815241 |
| 5 | 3.142   | 8.82747  |
| 6 | 3.1416  | 8.825023 |
| 7 | 3.14159 | 8.824962 |

**Fig. C1**

$2^x$  and  $x^2$  are increasing functions of  $x$  for  $x > 0$  and this allows us to deduce that  $\pi^2 > 2^\pi$ , as follows.

We know that  $\pi$  is between 3 and 3.142

25

$$\pi < 3.142 \Rightarrow 2^\pi < 2^{3.142} = 8.82747$$

$$\pi > 3 \Rightarrow \pi^2 > 3^2 = 9$$

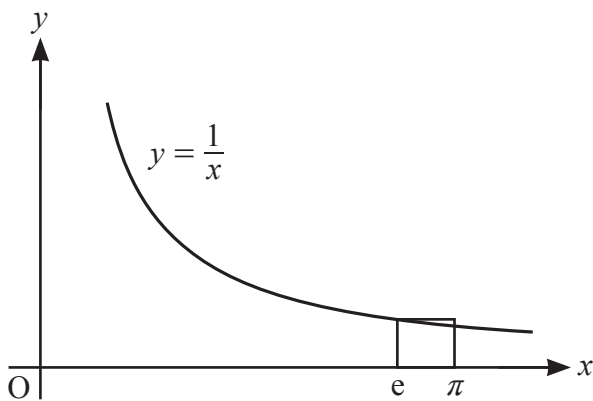
$$\text{So } \pi^2 > 9 > 8.82747 > 2^\pi$$

$$\text{Hence } \pi^2 > 2^\pi$$

**Which is bigger:  $\pi^e$  or  $e^\pi$ ?**

30

An indirect method, using calculus, enables us to prove that  $e^\pi$  is larger than  $\pi^e$ . Fig. C2 shows the curve  $y = \frac{1}{x}$  in the first quadrant together with the rectangle with vertices at the points  $(e, 0)$ ,  $(e, \frac{1}{e})$ ,  $(\pi, \frac{1}{e})$  and  $(\pi, 0)$ . We use the fact that the area under the curve between  $e$  and  $\pi$  is less than the area of this rectangle.



**Fig. C2**

The area of the rectangle is  $\frac{1}{e}(\pi - e)$

35

$$\int_e^\pi \frac{1}{x} dx < \frac{1}{e}(\pi - e)$$

$$\ln \pi - 1 < \frac{\pi}{e} - 1$$

$$\ln \pi < \frac{\pi}{e}$$

$e^x$  is an increasing function for all values of  $x$

$$\text{hence } \pi < e^{\frac{\pi}{e}}$$

40

Assuming that the usual rules of indices apply to irrational powers of irrational numbers, raising both sides of the inequality to the power  $e$  gives the desired result.

Using a similar method, it can be shown that  $e^a > a^e$  for any positive number  $a \neq e$ .

An alternative method for showing that  $e^a > a^e$  for any positive number  $a$  is to show that the only stationary point on the curve  $y = \frac{\ln x}{x}$  (a maximum) occurs where  $x = e$ .

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